Notes on the proof of direct sum for linear subspace

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Abstract. Linear space is an abstract concept in algebra, and results related to direct sum are complicated ones among the research of linear subspace. In this paper, two models for direction sum proof are proposed, and a typical proposition is introduced, and proved by using proposed model, so as to give a universally applicable proof method.

1. Introduction

Higher algebra is a fundamental and important course for polytechnic students [1], which helps to cultivate students' ability of abstract thinking and problem solving [2]. Though theorems related to linear space are comparatively difficult to grasp, research of linear space is worthwhile since it is a main part of this course and its application plays an important role in many aspects, including design of urban grand street [3], identification of Volterra kernels of nonlinearly dynamical systems [4], and linear space in campus [5].

Linear space is an important and abstract concept of higher algebra, and it possesses special property. For example, linear space has its own subspaces, definitions and operations like sum and intersection and it can be generated by other spaces mutually, and these proving details in sum and intersection of spaces are sophisticated that always make the proof harder. Fortunately, despite the abstraction of the concepts and the difficulties of the proof, there are regulations can be used to prove the sum especially the direct sum.

Many researchers utilized sorts of concrete methods (e.g., direct sum and peculiar matrix, direct sum and linear transformation, decomposition theorem of direct sum, direct sum and Cayley–Hamilton theorem) to solve problems of direct sum of linear space, but a few focused a basic model which has great significance.

Therefore, the purpose of this paper is to study the basic models of direct sum for linear space. By using these models, researchers can directly organize proper arguments to prove the existence of direct sum in certain circumstance. In this paper, we propose two basic models and give summaries to the common questions. In model I, since the first three equivalent propositions have been shown (see [6]), attention are paid to put forward new solutions to prove the fourth equivalent proposition [7, 8]. In model II, new methods and typical example are combined to prove the third combination [9].

The rest of the paper is organized as follows: Section 2 gives the preliminary discussion of the concept of direct sum for linear subspace. Section 3 proposes new solutions to prove the direct sum. Section 4 presents the analysis and proof of typical example. Section 5 provides the conclusion.

2. Traditional proof method of direct sum, 4 equivalence. (Model I)

Definition:

Let $W = V_1 + V_2 \subseteq V$ *, where* V_1 *,* V_2 *are subspaces of* V *. If decomposition of a vector* a *,* $a = a_1 + a_2$ *,* $a_i \in V_i$ $(i = 1,\ 2)$ *, is unique, then the sum is called the direct sum, denoted as* $W = V_1 \oplus V_2$ *[6].*

Proof of the equivalent proposition of $W = V_1 \oplus V_2$ forms model I.

Theorem 1 (Model I). *If* $W = V_1 + V_2$*, then the following four propositions [7, 8, 10] are equivalent to* $W = V_1 \oplus V_2$.

1. Decomposition like $a = a_1 + a_2$*,* $a_i \in V_i$ $(i = 1,\ 2)$ *of every vector* a *in linear space* W *is unique.*

2. $a_1 + a_2 = 0$*,* $a_i \in V_i$ *(*$i = 1,\ 2$*) can only be set up when* $a_i = 0$*, (*$i = 1,\ 2$*);*

3. $V_1 \cap V_2 = \{0\}$*;*

4. Vector with unique decomposition can be found in W*.*

Proof: Since the first three propositions have been proved to be equivalent see [6], only the equivalence of the 4-th proposition is needed. Two proofs are proposed to show the equivalence of the first and the fourth proposition.

Proof method I

(i) Condition $1 \Rightarrow$ condition 4: Suppose the decomposition is not unique. Since decomposition of every vector in linear space $V_1 + V_2$ is unique, 0 can also be divided as $0 = 0_1 + 0_2$ uniquely, where $0_1 \in V_1$, $0_2 \in V_2$. Therefore, vector with unique decomposition can be found in $V_1 + V_2$.

(ii) Condition 4 $\Rightarrow$ 1: Let $V_1 = L(a_1, a_2, \mathbf{L}, a_{K_1})$, $V_2 = L(b_1, b_2, \mathbf{L}, b_{K_2})$, $V_1 \cap V_2 = L(g_1, g_2, \mathbf{L}, g_{K_3})$, then for any $a = (a_{1,}\ \mathbf{L}, a_{K_1}, b_1, \mathbf{L}, b_{K_2}) \in (V_1 + V_2)$, we have $a = l_1 a_1 + \mathbf{L}\ + l_{k_1} a_{k_1} + l_{k_1+1} b_1 + \mathbf{L} + l_{k_1+k_2} b_{k_2}$. However, since $V_1 \cap V_2 \neq \{0\}$, $g_i = 0$ $(i \in \{1, 2, \mathbf{L}, k_3\})$. To simplify the process, let's assume $g_1 \neq 0$, then $a = (l_1 a_1 + l_2 a_2 + \mathbf{L} + l_{k_1} a_{k_1} + n g_1) +$ $(l_{k_1+1} b_1 + l_{k_1+2} b_2 + \mathbf{L} + l_{k_1+k_2} b_{k_2} - n g_1)$ $(n = 1,\ 2,\ \mathbf{L})$, where $(l_1 a_1 + l_2 a_2 + \mathbf{L} + l_{k_1} a_{k_1}\ + n g_1) \in V_1$ and $(l_{k_1+1} b_1 + l_{k_1+2} b_2 + \mathbf{L}\ + l_{k_1+k_2} b_{k_2}\ - n g_1)_2 \in V$. Furthermore, decomposition of a differs when n differs, in another word, decomposition of $a \in (V_1 + V_2)$ is not unique. Therefore, condition $4 \Rightarrow$ condition 3 as well as condition $4 \Rightarrow$ condition 1 are proved.

Thus the theorem follows.

Proof method II

(i) Condition $1 \Rightarrow$ condition 4: As the definition claims, decomposition of every vector in $V_1 + V_2$ is unique, therefore, vector with unique decomposition can be found in $V_1 + V_2$.

(ii) Condition $4 \Rightarrow$ condition 1: Suppose the decomposition is not unique. Let $V_1 = L(a_1, a_2, \mathbf{L}, a_{K_1})$, $V_2 = L(b_1,\ b_2, \mathbf{L}, b_{K_2})$, then $a = l_1 a_1 + l_2 a_2 +\ \mathbf{L} + l_{k_1}$, so $a_{k_1} + l_{k_1+1} b_1 + l_{k_1+2} b_2\ + \mathbf{L} + l_{k_1+k_2} b_{k_2}$ maps to $a \in (V_1 + V_2)\ = L(a_1, \mathbf{L}, a_{k_1},\ b_1, \mathbf{L}, b_{k_2})$. Since the decomposition of vector 0 is not unique, $a = (l_1 a_1 + l_2 a_2 + \mathbf{L} + l_{k_1} a_{k_1} + g_1) +\ (l_{k_1+1} b_1 + l_{k_1+2} b_2 + \mathbf{L} + l_{k_1+k_2} b_{k_2} + g_2)$ is not unique, where g_1, g_2 satisfy $0 = 0 + 0 = g_1 + g_2$, ($g_1 \neq 0$ or $g_2 \neq 0$ and $g_1 \in V_1$, $g_2 \in V_2$). Therefore, condition $4 \Rightarrow$ condition 1 is proved.

Thus the theorem is proven.

Note:

Several notes are obtained from the above discussion.

1) Direct sum: Decomposition of every vector in sum space is unique $\Leftrightarrow$ Vector with unique decomposition can be found in sum space $\Leftrightarrow$ Decomposition of 0 is unique.

2) Decomposition of 0 in sum space is unique $\Leftrightarrow$ $V_1 \cap V_2 = \{0\}$.

3. New ideas for proof of direct sum (Model II)

The new idea of direct sum proof, $V_1 \oplus V_2 = V$ (V_1, V_2 are subspaces of V) is given in this section, denoted as model II. In order to clarify that what conditions are needed to prove $V_1 \oplus V_2 = V$ if V_1, V_2 are subspaces of V ,the following three conditions are straightforward..

①$V_1 + V_2 = V$

② $\dim(V_1) + \dim(V_2) = \dim(V)$

③$V_1 \cap V_2 = \{0\}$

Theorem 2 (Model II)*. Combination of any two of above conditions suffices to prove* $V_1 \oplus V_2 = V$ *.*

Proof: First we consider combination 1: $V_1 + V_2 = V$ and $\dim(V_1) + \dim(V_2) = \dim(V)$. Compared with Dimension formula, $\dim(V_1 \cap V_2) = 0$ can get from $\dim(V_1 + V_2) + \dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2)$, $V_1 \cap V_2 = \{0\}$. Furthermore, $V_1 + V_2 = V$. Therefore, $V_1 \oplus V_2 = V$.

For combination 2: $V_1 + V_2 = V$ and $V_1 \cap V_2 = \{0\}$. From the third equivalent condition, it's easy to get $V_1 \oplus V_2 = V$.

Finally, in combination 3: $\dim(V_1) + \dim(V_2) = \dim(V)$ and $V_1 \cap V_2 = \{0\}$ [9], suppose $r(V) = n$, and $a_1, a_2, \mathbf{L}, a_{k_1}$ is basis of V_1, $b_1, b_2, \mathbf{L}, b_{K_2}$ is basis of V_2, and $g \in V_1 \cap V_2$, g can be linear represented as $g = l_1 a_1 + l_2 a_2 + \mathbf{L} + l_{k_1} a_{k_1} = m_1 b_1 + m_2 b_2 + \mathbf{L} + m_{k_2} b_{k_2}$. Since $V_1 \cap V_2 = \{0\}$, it is easy to obtain that $l_1 a_1 + l_2 a_2 + \mathbf{L}\, l_{k_1} a_{k_1} = 0$ and also we have $m_1 b_1 + m_2 b_2 + \mathbf{L} + l_{k_2} b_{k_2} = 0$. Now $a_1, a_2, \mathbf{L}, a_{k_1}$ and $b_1, b_2, \mathbf{L}, b_{k_2}$ are linearly independent, therefore $l_1 = l_2 = \mathbf{L} = l_{k_1} = m_1 = \mathbf{L} = m_{k_2} = 0$, and $a_1, a_2, \mathbf{L}, a_{k_1}, b_1, b_2, \mathbf{L}, b_{k_2}$ are linearly independent, which means that $a_1, a_2, \mathbf{L}, a_{k_1}, b_1, b_2, \mathbf{L}, b_{k_2}$ is basis of V. So, $V_1 + V_2 = V$. With consideration of $V_1 \cap V_2 = \{0\}$, we obtain $V_1 \oplus V_2 = V$.

4. Application: Put the two models into proving a typical proposition

4.1 Proposition

Pleas show that $L\left(A^{+}A\right) \oplus L\left(E - A^{+}A\right) = R^{n}$*, where* A^{+} *(*$A^{+} \in R^{n\times m}$*) is the generalized inverse of* A *(*$A \in R^{m\times n}$*), which satisfies* $AA^{+}A = A$ *and* $A^{+}AA^{+} = A^{+}$*,* $L(A)$ *is linear space generated from the column vectors of* A*.*

4.2 *Lemma.*

If $A \in P^{n\times n}$*,* $B \in P^{n\times n}$*, then we have* $r(A+B) \le r(A) + r(B)$*. [6]*

By using theorem 2 (model II), this problem can be solved by proving the three conditions.

4.3 Proof of condition ①

4.3.1 Proof method I

For any $X \in R^{n}$, denoted $X = EX = \left(A^{+}A + E - A^{+}A\right)X = \left(A^{+}A\right)X + (E - A^{+}A)X$, which shows that X is linear represented by the column vectors of $A^{+}A$ and $E - A^{+}A$. Therefore, we have $L\left(A^{+}A\right) + L\left(E - A^{+}A\right) = R^{n}$.

4.3.2 Proof method II

Suppose $A^+A=(a_1,a_2,\cdots,a_n)$, $E-A^+A=(e_1-a_1,e_2-a_2,\cdots,e_n-a_n)$, where $e_1,e_2,\cdots,e_n$ is a standard basis of R^n. So $L(A^+A)+L(E-A^+A)=$ $L(a_1,a_2,\cdots,a_n,\ e_1-a_1,e_2-a_2,\ \cdots,e_n-a_n)$. It shows that standard basis of R^n is linear represented by $a_1,a_2,\cdots,a_n,e_1-a_1,e_2-a_2,\cdots,\ e_n-a_n$, which means $L(A^+A)+$ $L(E-A^+A)=R^n$.

4.3.3 Proof method III

Let $\dim(L(A^+A,\ E-A^+A)=m$, then $m=r(A^+A,\ E-A^+A)\ =r(A^+A,E)$. While combined with $A^+A\in R^{n\times n}$, $E-A^+A\in R^{n\times n}$ and the result $r(A^+A,\ E-A^+A)=n$, we get $r(A^+A,E-A^+A)=n$, $m=n$. Hence, n linearly independent vectors in $L(A^+A)+L(E-A^+A)$ form a basis of R^n, then it suffices to show that $L(A^+A)+L(E-A^+A)=R^n$.

4.4 Proof of condition ②

4.4.1 Proof method I

Note that $A^+AA^+=A^+$, then we have the result $(E-A^+A)A^+A=0$. Since $A^+A=(a_1,\ a_2,\ \cdots,\ a_n)$, $a_1,a_2,\cdots,a_n$ can be considered as the solution of $(E-A^+A)X=0$, and it ensures that $r(A^+A)+r(E-A^+A)\le n$. But we have $r(E-A^+A+A^+A)\le$ $r(A^+A)+r(E-A^+A)$ according to the Lemma.

Therefore, all of the deduction show that $r(A^+A)+r(E-A^+A)=n$, and $\dim(L(E-A^+A))+\dim(L(A^+A))=n$.

4.4.1 Proof method II

It is easy to show that $r(A^+A)+r(E-A^+A)=$ $r\begin{pmatrix} A^+A & 0 \\ 0 & E-A^+A \end{pmatrix}=r\begin{pmatrix} A^+A & A^+A \\ 0 & E-A^+A \end{pmatrix}$ $=r\begin{pmatrix} 0 & A^+A \\ 0 & E-A^+A \end{pmatrix}$ $=r\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}$, suffices to say $r(A^+A)+r(E-A^+A)=n$, and $\dim(L(E-A^+A))+\dim(L(A^+A))=n$

4.5 Proof for condition ③

4.5.1 Proof method I

Now let $A^+A=(a_1,a_2,\cdots,a_n)$ and $(E-A^+A)=(b_1,b_2,\cdots,b_n)$. If there exists a vector a in R^n that satisfies $a\in L(A^+A)$ and $\cap\,(E-A^+A)$, then $a=A^+AX_1=(E-A^+A)X_2$.

Times A^+A to the both sides of $A^+AX_1=(E-A^+A)X_2$, then we have the equation $A^+A(A^+AX_1)=A^+A\ (E-A^+A)X_2=0$. However, we have $A^+A(A^+AX_1)=A^+AA^+AX_1=$ A^+AX_1. Therefore, $A^+AX_1=0$ and $L(A^+A)\cap L(E-A^+A)=\{0\}$.

4.5.1 Proof method II

Similarly as former section, let $A^+A=(a_1,a_2,\cdots,a_n)$, and $(E-A^+A)=(b_1,b_2,\cdots,b_n)$. If vector a that satisfies $a\in L(A^+A)$ and $a\in L(E-A^+A)$ can be found in R^n, then $a=A^+AX_1=(E-A^+A)X_2$.

Based on the definition of $A^+AA^+ = A^+$, we have $A^+AA^+A = A^+A$, which yields to $\left(E - A^+A\right)A^+A = 0$ and $(E - A^+A)A^+AX_1 = 0$. That suffices to $\left(E - A^+A\right)a = 0$ **L L** (1)

Similarly, we have

$A^+A\left(\left(E - A^+A\right)X_2\right) = 0$ and $A^+Aa = 0$ **L L** (2)

Combined (1) with (2), we have $a = 0$. So $L\left(A^+A\right) \cap L\left(E - A^+A\right) = \{0\}$.

In all, any two proofs of the three conditions are sufficient to prove the proposition $L(A^+A) \oplus L(E - A^+A) = R^n$, and there are at least sixteen differential methods.

5. Conclusions

As shown in this paper, there are three directions to prove the direct sum for linear space. (1). Macroscopic view: To prove the direct sum is to prove sum of dimensions equals to dimension of sum of spaces. (2). Microscopic view: The task is to prove the common elements of the two subspaces is 0. (3). Diversity view: When proving from the view of dimension, it's feasible to use generated subspace, matrix and equivalent base.

When proving the intersection of two spaces is 0, we introduced the system of linear equation, rather than the commonly used methods, e.g., reduction to absurdity, basis of linear space. Furthermore, this paper presents strategies to common direct sum problems from the view of concept of direct sum, and since the relationship between the two subspaces and the whole space is representative, more attention are paid to the relationship between them. The same goes well for problems with more than two spaces. However, rang and nucleus are two special subspaces, where rang limits the size of the space under transformation, and nucleus defines the corresponding element to the null element. Therefore, the two subspaces satisfy the requirement that sum of dimensions equals to the dimension of the sum space, which means the extra task is to prove one of the other two conditions mentioned in model II when proving they are the two subspaces of the direct sum.

Research of decomposition of direct sum has significant importance in transforming problems with high dimension to problems with lower dimension. For example, to simplify the process, two dimensional vector problems could be converted to problems in two orthogonal complement spaces with one dimension by using the properties of orthogonal complement space and dimensions of lower dimension subspace. Besides, it's theoretical feasible to divide the direct sum, and generating a generation space by combining bases of one space randomly is a good example. However, since spaces related to problems are always monstrous, proof of direct sum is harder. But, fortunately, base forms the space, and reflects the relationship between subspaces and sum space. Therefore, we can solve direct sum problems by proving bases of subspaces are linearly independent and the number of them equals to the number of the base of the sum space.

In this paper, model I was set up under the premise of $V = V_1 + V_2$, and model II was set up under the premise of V_1, V_2 is the linear subspaces of V. When compared the two models, model I is a simple proof model, and model II is a comparatively complicated one. But both of them have covered the common proof models of direct sum, including the abstruse models.

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References

[1] Xiaoping Ye. Cognition and practice of reformation of linear algebra teaching method. Journal of Sun Yat-Sen University,1998), 2: 128-130.

[2] Lijie Ma. Notes of education of linear algebra. Reform and Open 12 (2009), Reform and opening, 2009, 12:198-199.

[3] Fuzhi Xue, FangHu. An Exploration of Linear Spatial Planning and Design of Urban Grand Street—The Case of the Three Trunk Roads in Shenzhen. Journal of Urban Planning, 2010, 192 (7): 60-65.

[4] Wei Si, Zhenmin Duan, Tao Hai. Novel method based on projection of vectors in linear space to identify Volterra kernels of arbitrary orders. Application Research of Computers, 2008, 25 (11): 3340-3342.

[5] Yan Tao. Linear Space Composition of Campus Environment — Taking southern campus of East China Jiaotong University as xample. Journal of East China Jiaotong University, 2011, 28 (4): 65-69.

[6] Department of mathematic of Peking university. Higher algebra, Higher Education Press, 2003.

[7] Zonming Sun. Equivalent conditions of direct sum for linear subspace. Journal of Nanchon Normal College, 1988, 9 (1): 1-1.

[8] Zonming Sun, Zhenguo Li, Menchang Mei. Equivalent conditions of dimension formula and direct sum for subspace. Journal of Changsha University, 1998, 2: 47-50.

[9] Qin Yang. Proof of direct sum of subspaces. Journal of Fuyang Normal College, 2009, 26 (4): 27-28..

[10] Ying Liu, Dongli Liu. Discussion of direct sum for linear space. Journal of Science of Teachers' College and University 1. 2011, 31 (1): 40-45.